The Cobb-Douglas Production Function and Political Economy

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Université Paris X – Nanterre, April 28, 2009

Abstract: In classical and Marxian political economy economic advancement takes place in stages of development. These stages are characterized by different functional distributions of income. In this paper we show that the Cobb-Douglas Production Function is the only production function which has the property of a constant functional distribution of income of the factors of production. Furthermore it is argued that an aggregate production function is not appropriate to analysing macro-economic processes of a capitalist economy. But it is a very useful tool in the hands of a socialist planning authority.

Keywords: political economy; classical economics, Marxian economics; socialist economics; aggregate production function; functional distribution of income; Cobb-Douglas Production function; stages of economic development

JEL Codes: B12; B14; B51; E25; O11; 020; P16; P21

Introduction

David Ricardo defines the principle task of Political Economy as to determine the laws which govern the distribution of income amongst the social classes: “The produce of the earth — all that is derived from its surface by the united application of labour, machinery, and capital, is divided among three classes of the community; namely, the proprietor of the land, the owner of the stock or capital necessary for its cultivation, and the labourers by whose industry it is cultivated. But in different stages of society, the proportions of the whole produce of the earth which will be allotted to each of these classes, under the names of rent, profit, and wages, will be essentially different; depending mainly on the actual fertility of the soil, on the accumulation of capital and population, and on the skill, ingenuity, and instruments employed in agriculture. To determine the laws which regulate this distribution, is the principal problem in Political Economy: much as the science has been improved by the writings of Turgot, Stuart, Smith, Say, Sismondi, and others, they afford very little satisfactory information respecting the natural course of rent, profit, and wages.” (Ricardo, 1951, p. Preface)

Ricardo takes it for granted that human society consists of social classes and we find with him the idea that history of society develops in stages and that these stages are characterized by different functional distributions of income amongst the classes. These are obviously the bricks Marx and Engels have used to build their theory of historical materialism.
The Production Function and the Distribution of Income

In modern economic theory the functional distribution of income is based on the theory of production which uses the concept of the production function. Now, if there can be defined an aggregate production function explaining the functional distribution of income we would be interested in a function which generates a distribution of income which is constant as long as society's development is confined to a specific stage.

The production function which fulfills these requirements is the Cobb-Douglas Production Function, named after its inventors Cobb and Douglas (Cobb & Douglas, 1928). Although it has been found that the concept was already known to Knut Wicksell. In the following we shall show that the Cobb-Douglas function is the only function which fulfills the requirements of yielding a constant functional distribution of income. The mathematical form of the function is stated as output \( Q \) being a function of the factors of production capital, \( K \), and labour, \( L \), only. The value of output, \( pQ \), is national income, \( Y \). One should observe that the factor land is commonly omitted as land is treated as capital and there is no social class of landlords in the capitalist stage of society. The functional form is as follows

\[
Q = K^a L^\beta
\]

and

\[
Y = pQ
\]

where \( Q \) is output, \( K \) stands for capital and \( L \) for labour. \( p \) is the price and \( Y \) is income. \( a \) and \( 1-a \) are the production elasticities for capital and labour respectively. The production elasticity is defined as the ratio of the marginal productivity to the average product of a factor of production.

When we divide the marginal productivity of capital, \( \frac{\delta Q}{\delta K} \) by \( Q/K \) we have

\[
\frac{\delta Q}{Q} = aK^{a-1} L^\beta \frac{K}{Q} = a
\]

and equally for labour we divide the marginal productivity of labour \( \frac{\delta Q}{\delta L} \) by the average productivity of labour \( Q/L \) and obtain

\[
\frac{\delta Q}{Q} = \beta K^a \frac{L^\beta}{L} = \beta
\]

Another characteristics is, if \( a + \beta = 1 \), this function has constant returns to scale. A proportional increase \( \lambda \) of all the factors of production yields an equal proportional increase of output.

\[
(\lambda K)^a (\lambda L)^\beta = \lambda^a K^a \lambda^\beta L^\beta = \lambda^{a+\beta} K^a L^\beta
\]

If \( a + \beta = 1 \)

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Furthermore, under perfect competition and profit maximization of the firms the wage rate, \( w \), equals the value of the marginal product of labour \( w = p \frac{\delta Q}{\delta L} \) and the rate of interest, \( r \), equals the value of the marginal product of capital \( r = p \frac{\delta Q}{\delta K} \).

The price, \( p \), is taken as numéraire, \( p = 1 \), and is omitted from the equations.

So we can simplify \( Y = Q \) and \( w = \frac{\delta Q}{\delta L} \) and \( r = \frac{\delta Q}{\delta K} \).

From \( 1 = a + \beta \) we get \( 1 = a + \beta = r K/Q + w L/Q \)

\[ Q = r K + w L = Y \]

Under perfect competition and constant returns to scale the functional distribution of income amongst the factors of production is equal to the sum of the values of the marginal products of these factors.

\[ Y = p Q = p \frac{\delta Q}{\delta K} K + p \frac{\delta Q}{\delta L} L \]

**Constant Factor Shares and the Cobb-Douglas Production Function**

It remains to be shown that if the functional distribution of income is constant then the production function is a Cobb-Douglas production function with constant returns to scale.

In order to show this it is convenient to state the function in its implicit form. We express output per labour unit as a function of the capital/labour ratio.

If \( 1 = a + \beta \) then \( \beta = 1 - a \) and we may write \( Q = K^a L^{1-a} \)

\[ Q/L = q = K^a L^{-a} = \frac{K^a}{L^a} = \left( \frac{K}{L} \right)^a = k^a \]

\[ q = k^a \]

Output per labour unit, \( q \), is a function of the capital labour ratio, \( k \). Notice that this function is homogeneous of degree 0. A proportional increase of both factors of production leaves \( k \) unchanged and therefore output per labour unit, \( q \), also.
Now we start from the assumption of constant shares of labour and capital in income.

Income is the sum of profits, $P$ and wages, $W$.

$$ Y = P + W $$

Profit, $P$ equals the returns on capital, $rK$, and wages equals the returns to labour, $wL$.

Substituted into the income equation gives

$$ Y = rK + wL $$

Taking again $Y = Q$ we restate it as

$$ Q = rK + wL $$

Now we assume profits to be a constant share of output $a = (rK)/Q$ or $a = P/Y$

$$ a \frac{Q}{rK} $$

We obtain the relation in terms of output per labour unit by dividing through $L$.

$$ aq = rk, \text{ where } q = Q/L \text{ and } k = K/L. $$

We assume as above that under perfect competition $r = \frac{\delta Q}{\delta K}$ and this is equal to $dq/dk$.

$$ aq = dq/dk k $$

By rearranging we have

$$ a \frac{dk}{k} = dq/q $$

or

$$ dq/q = a \frac{dk}{k}, $$

And further we can write

$$ d \ln(q) = a * d \ln(k). $$

and this integrated is

$$ \ln(q) = a * \ln(k) $$

and therefore

$$ q = k^a $$

which is the Cobb-Douglas production function.

This is a very important result. If the functional distribution of income is constant over time and if there is perfect competition than one should observe a relationship between average labour productivity $q$ and the capital labour ratio $k$ which is of type Cobb-Douglas. However the derivation of this result reveals serious limitations.
First we have assumed \( p \) to be a scalar and have used it as a numéraire in order to have an equality between income and output. But this implies that it would be possible to aggregate all commodities of an economy into one type of commodity which represents output, \( Q \), as well as the capital input, \( K \). Allen makes this assumption explicitly when he derives the result as stated above (Allen, Roy, 1968, p. 38 ff.). This is known to be the aggregation problem. In fact, a technical production function is of a form \( q = f(x_1, x_2, \ldots, x_n) \) where the \( x_i \) represent different quantities of types of inputs. When we substitute them by one scalar \( K \) this implies that we have valued these inputs which requires a set of prices or a price vector. \( K \) must be treated as a value expression and the same is true for output \( Q \).

**The Aggregate Production Function and the Labour Theory of Value**

Let's assume that the labour theory of value holds. In this case we would be able to aggregate all the material inputs to some amount of labour value which represents capital and equally we can aggregate all outputs produced to some amount of labour values. The dimensions of this function would be for both sides of the equation labour units. The production function would not be a technical but an economic production function.

In his discussion of historical materialism and the labour theory of value Samir Amin puts forward such a vision where labour values are supposed to be independent of distribution (Amin, Samir, 1977, pp. 7-8) Unfortunately his argumentation is limited only to the case of linear production functions. For production functions of a substitutitional type as is the Cobb-Douglas function the combination of factor inputs changes with a change in the distributional variables \( r \) and \( w \) and therefore the labour values of outputs also changes as these variables change. Here seems to be a circularity argument involved. However this is not so. For any period of production one might take the amounts of factors of production in their physical form as given and fully employed. Then there exists a general equilibrium where the values of \( w, r \) as well as of \( Q \) and \( K \) are determined. But this solution depends also on demand and there are then for all commodities particular production functions. Only under very restrictive conditions one can determine an aggregate production function also, for example if the organic compositions of capital are all equal in all sectors of the economy, a condition for which the Marxian labour theory of value would also hold (Garegnani, 1970).

**The Problem of an Aggregate Production Function**

This is not the only problem involved with using an aggregate production function in determining the functional distribution of income.

We shall point out here only a secondary aspect. In the formulation above, output is limited to value added only. But production comprises not only value added but also the replacement of the circulating capital or intermediary production as well. Only if one includes this category and expands the concept of output to gross production one may compare the results of marginal analysis with those of the analysis of an equilibrium situation in terms of linear production systems.
Fisher and Felipe (Felipe & Fisher, 2003) criticize strongly the use of the aggregate production function in growth theory. But one should realize that the theory of economic growth is concerned with finding the conditions of an equilibrium growth path and so the use of a Cobb-Douglas production function is almost a necessity as only this function guarantees the constancy of the distribution of income. In the context of a planned socialist economy the concept is extremely useful. Kantorovich makes extensive use of the aggregate production function. “The subject of our paper is also a one-product model. At first glance a one-product model is an extreme stage of abstraction in relation to the real national economy. Only one product is considered: it is eaten, it is used for clothing, it is used to construct factories and to make machines. In fact this abstraction is not so great in actuality, and under an economic approach all products may be commensurated – in money or in labor (in accordance with the adopted conception of price formation). We recall that Marx's two-product model of expanded reproduction is the same kind of abstraction, but it nevertheless provides extensive possibilities for meaningful analysis of the economy. Valuing each product in the same units, we can consider any model of the national economy as a one-product model, and from the mathematical viewpoint the method by which all products may be commensurated and transformed into one – through money, labor, energy, or any other method – is not so important.” (Kantorovich & Vainshtein, 1976, p. 72).

Conclusions

Concluding we may state that the Cobb-Douglas production function in its aggregate form when it is empirically valid represents a system of production which is in a state of stability from a Ricardian point of view. However, it seems not to be possible to reduce the distribution of income amongst the classes of society, which is an essential part of the class struggle, to a simple mathematical function, in particular to discuss changes of the level of employment of the factors of production as results of changes of the prices of the services of the factors of production. On the other hand the Cobb-Douglas production function is a very valuable tool in the hands of the socialist economic planner.


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Bibliography


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